Drawdowns, Drawups, their joint distributions, detection and financial risk management

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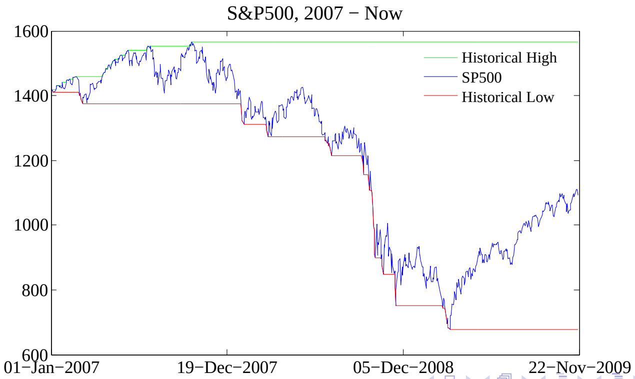
Motivation

- The price of a stock or index is fluctuate, and may have a big drop or a big rally over a period $[0, T]$.
  - The present decrease from the historical high
  - The present increase over the historical low



Mathematical Definitions

- A stochastic process $\{X_t; t \geq 0\}$.
- Its drawdown and drawup processes.

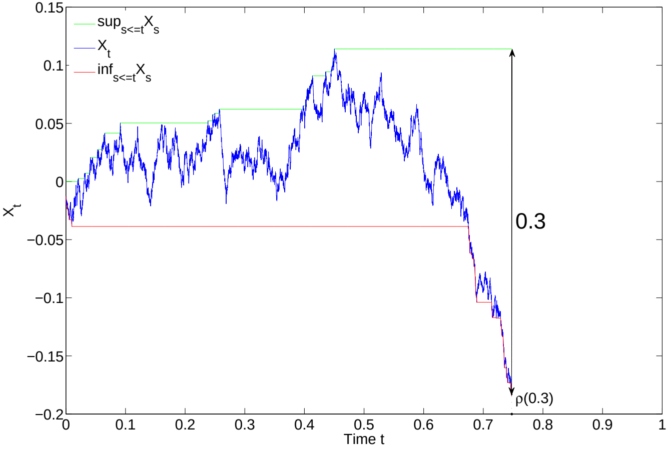
$$DD_t = \sup_{s \leq t} X_s - X_t, \; DU_t = X_t - \inf_{s \leq t} X_s.$$

- Drawdowns and drawups.

$$\begin{aligned} T_D(a) &= \inf\{t \geq 0 | DD_t \geq a\}, \\ T_U(b) &= \inf\{t \geq 0 | DU_t \geq b\}. \end{aligned}$$

The goal: characterize the probability $P_x(T_D(a) \leq T_U(b) \wedge T)$.

The first range time $\rho(a) = T_D(a) \wedge T_U(a)$



Probability distribution

- On $\{T_D(a) \le T_U(a)\}$, $X_{T_D(a)} \in [-a,0)$.
- It suffices to determine

$$P_0(T_D(a) \in dt, T_U(a) > t, X_t \in du), \; -a \le u < 0.$$

- Connection with the hitting probability

$$\begin{aligned} & P_0(T_D(a) \in dt, T_U(a) > t, X_t \in du) \\ = \; & P_0(\tau_u \in dt, \sup_{s \le t} X_s \in du + a) \\ = \; & \frac{\partial}{\partial a} P_0(\tau_u \in dt, \sup_{s \le t} X_s < u + a) du \\ = \; & \frac{\partial}{\partial a} P_0(\tau_u \in dt, \tau_{u+a} > t) du. \end{aligned}$$

- We have a closed-form formula for $P_0(T_D(a) \le T_U(a) \wedge T)$ under drifted Brownian motion dynamics. (RW is similar)

Laplace transform under general diffusion dynamics: $a = b$

- Consider a linear diffusion X on $I = (l, r)$ with continuous generator coefficients and natural (or entrance) boundaries.
- The goal: Laplace transform $E_x\left\{e^{-\lambda T_D(a)} \cdot \mathbb{1}_{\{T_D(a) < T_U(a)\}}\right\}$.
- For $-a + x \le u < x$, $\lambda > 0$ with $X_0 = x$,

$$\begin{aligned} & L_x^X(\lambda, u; a, a)du \\ \equiv\ & E_x\left\{e^{-\lambda T_D(a)} \cdot \mathbb{1}_{\{T_D(a) < T_U(a), X_{T_D(a)} \in du\}}\right\} \\ =\ & \frac{\partial}{\partial a} E_x\left\{e^{-\lambda \tau_u} \cdot \mathbb{1}_{\{\sup_{s \le \tau_u} X_s < u + a\}}\right\} du \\ =\ & \frac{\partial}{\partial a} E_x\left\{e^{-\lambda \tau_u} \cdot \mathbb{1}_{\{\tau_u < \tau_{u+a}\}}\right\} du. \end{aligned}$$

- The last conditioned Laplace transform of first hitting time is known through solutions of an ODE.

The conditioned Laplace transform of first hitting time

- Consider the SDE governing the linear diffusion X

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ X_0 = x.$$

- For $l < L \leq x \leq H < r$ and $\lambda > 0$, (Lehoczky 77')

$$E_x\{e^{-\lambda\tau_L} \cdot \mathbb{1}_{\{\tau_L < \tau_H\}}\} = \frac{g^\lambda(x)h^\lambda(H) - g^\lambda(H)h^\lambda(x)}{g^\lambda(L)h^\lambda(H) - g^\lambda(H)h^\lambda(L)},$$

where g^λ and h^λ are any two independent solutions of the ODE

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2} + \mu(x)\frac{\partial f}{\partial x} = \lambda f.$$

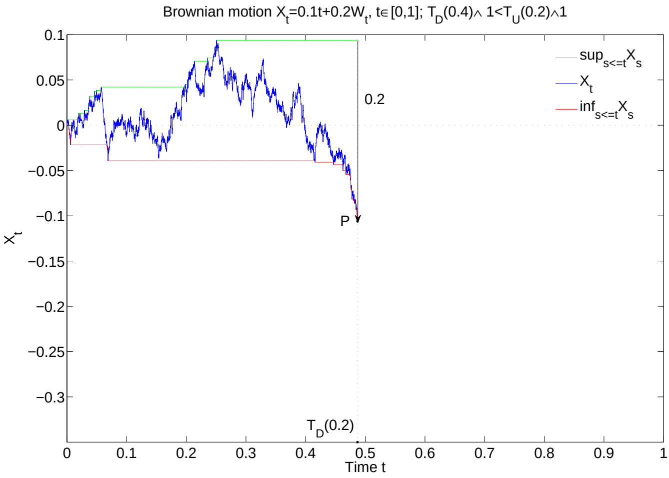
- For constant parameter case (X is a drifted Brownian motion), g^λ and h^λ are exponential functions.

Path decomposition

- If $a > b$, the strong Markov property of linear diffusion facilitate the use of Laplace transform and path decomposition.
- For any path in $\{T_D(a) < T_U(b)\}$
    1. $\{X_t; 0 \leq t \leq T_D(b)\} \sim$ Range process.
    2. $\{X_{t+T_D(b)} - X_{T_D(b)}; 0 \leq t \leq T_D(a) - T_D(b)\} \sim$ Hitting time with drawup constraint.

$$\Rightarrow T_D(a) \quad = \quad T_D(b) + \tau_{X_{T_D(b)}+b-a} \circ \theta_{T_D(b)}.$$

Illustration of a Brownian sample path with $a = 0.4, b = 0.2$



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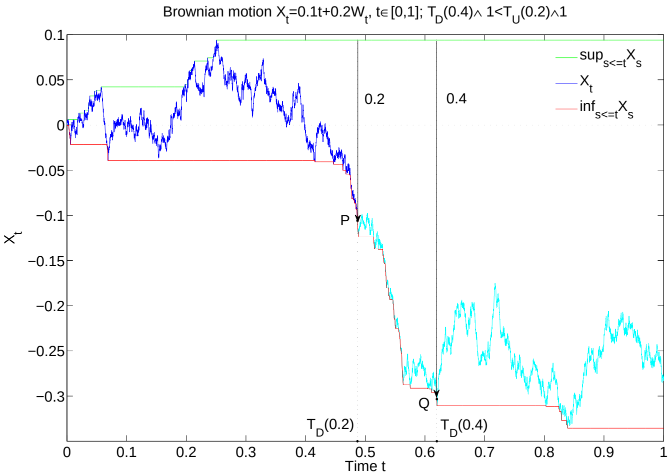


Laplace transform $a > b$

- Recall that on $\{T_D(a) < T_U(b)\}$ with $a > b$,

$$T_D(a) = T_D(b) + \tau_{X_{T_D(b)}+b-a} \circ \theta_{T_D(b)}$$

- Conditioning on $\{X_{T_D(b)} = u\}$,

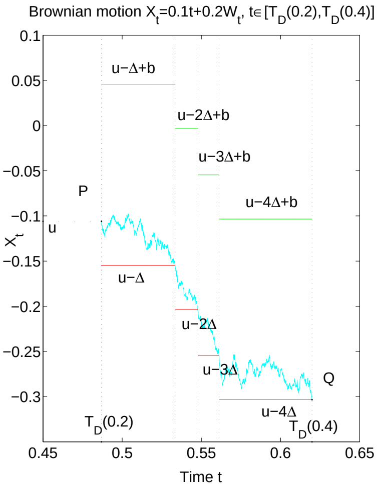
$$\begin{aligned} & E_x\{e^{-\lambda \tau_{u+b-a}\circ\theta_{T_D(b)}} \cdot \mathbb{I}_{\{\tau_{u+b-a}\circ\theta_{T_D(b)} < T_U(b)\circ\theta_{T_D(b)}\}} | X_{T_D(b)} = u\} \\ = \quad & E_u\{e^{-\lambda \tau_{u+b-a}} \cdot \mathbb{I}_{\{\tau_{u+b-a} < T_U(b)\}}\}. \end{aligned}$$

- For $-a + x < u < x$, $\lambda > 0$ with $X_0 = x$,

$$\begin{aligned} & L_x^X(\lambda, u; a, b) du \\ \equiv \quad & E_x\{e^{-\lambda T_D(a)} \cdot \mathbb{I}_{\{T_D(a) < T_U(b), X_{T_D(b)} \in du\}}\} \\ = \quad & L_x^X(\lambda, u; b, b) \cdot \underbrace{E_u\{e^{-\lambda \tau_{u+b-a}} \cdot \mathbb{I}_{\{\tau_{u+b-a} < T_U(b)\}}\}}_{\text{The strong Markov property}} du. \end{aligned}$$

The strong Markov property and discrete approximation

- Conditioning on $\{X_{T_D(b)} = u\}$, partition the interval $[u-a+b, u]$ into n subintervals with equal length $\Delta = (a-b)/n$.
- Use conditioned hitting times to approximate.
- Pass to the limit. The continuity of the sample path and bounded convergence theorem justifies this.



Path decomposition

- Relationship between Laplace transforms

$$\begin{aligned} & E_x\left\{e^{-\lambda T_D(a)} \cdot \mathbb{I}_{\{T_D(a)<T_U(b)\}}\right\} \\ = \quad & E_x\left\{e^{-\lambda T_D(a)}\right\} - E_x\left\{e^{-\lambda T_D(a)} \cdot \mathbb{I}_{\{T_D(a)>T_U(b)\}}\right\} \end{aligned}$$

- To get the very last Laplace transform, observe that on $\{T_D(a) > T_U(b)\}$,

$$T_D(a) = T_U(b) + T_D(a) \circ \theta_{T_U(b)}.$$

- Using strong Markov property of linear diffusion

$$E_x\left\{e^{-\lambda T_D(a)\circ\theta_{T_U(b)}} | X_{T_U(b)}\right\} = E_{X_{T_U(b)}}\left\{e^{-\lambda T_D(a)}\right\}.$$

- We can use reflection to find

$$E_x\left\{e^{-\lambda T_U(b)} \cdot \mathbb{I}_{\{T_D(a)>T_U(b), X_{T_U(b)}\in du\}}\right\}.$$

Illustration of a Brownian sample path with $a = 0.15, b = 0.3$

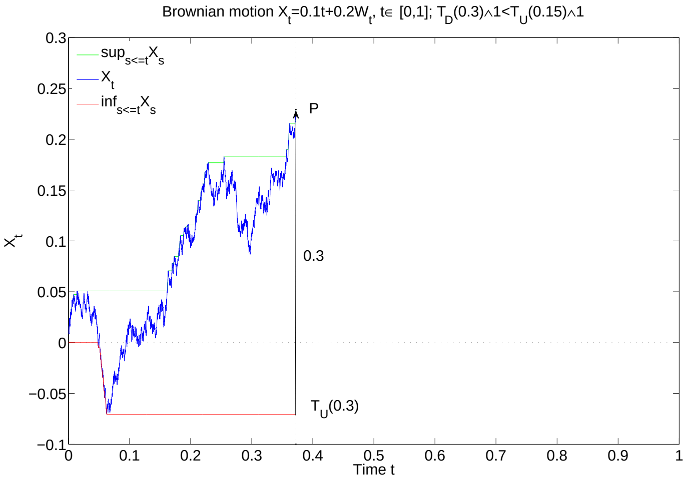
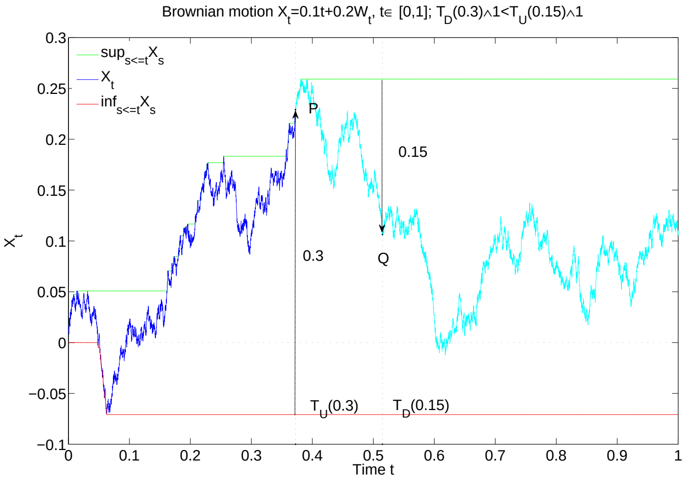


Illustration of a Brownian sample path with $a = 0.15, b = 0.3$



Detection of two-sided alternatives

- We sequentially observe a process $\{\xi_t\}$ with the following dynamics:

$$dX_t = \begin{cases} dw_t & t < \tau \\ \begin{array}{c} \alpha(X_t)dt + \sigma(X_t)dw_t \\ \text{or} \\ -\alpha(X_t)dt + \sigma(X_t)dw_t \end{array} & T \geq t \geq \tau \end{cases}$$

- probability of misidentification

$$\begin{aligned} P_x^{0,+}(T_D(a) < T_U(b) \wedge T) &= \int_0^\infty P_x^{0,+}(T_D(a) < T_U(b) \wedge t) \cdot \lambda e^{-\lambda t} dt \\ &= \int_0^\infty e^{-\lambda t} P_x^{0,+}(T_D(a) \in dt, T_U(b) > t) dt \\ &= L_x^{X^{0,+}}(\lambda; a, b), \qquad (1) \end{aligned}$$

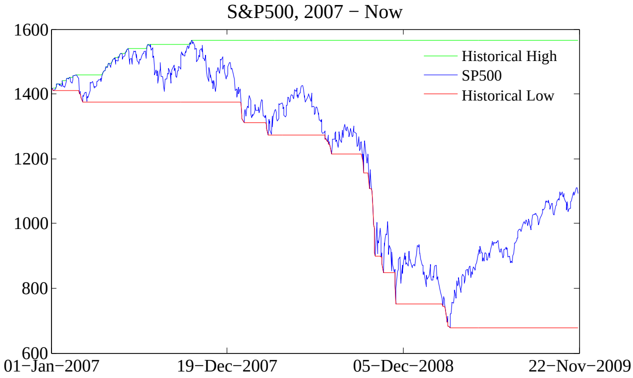
Detection of two-sided alternatives(cont)

- aggregate probability of misidentification

$$
\begin{aligned}
& \int P_y^{\tau,+}(T_D(a)\circ\theta(\tau) < T_U(b)\circ\theta(\tau)\wedge T) f_{X_\tau}(y|x)dy \qquad (2) \\
= & \int L_y^{X^{0,+}}(\lambda, a, b) f_{X_\tau}(y|x)dy,
\end{aligned}
$$

Digital call insurance

- Financial assets are risky.



- A digital call that pays $\mathbb{1}_{\{T_D(K) \leq T_U(K) \wedge T\}}$ can be perceived as an insurance against adverse movement in the market.

Pricing and replication

- The previously defined digital call only pays out one dollar (compensation) if the price process X draws down by K dollars before it draws up by the equal amount.
- Under no transaction cost and no arbitrage, the price of an option with payment at time T is just the expectation of the discounted cashflow in the future.
  - Let $B_t(T)$ be the price of a bond maturing at T, consider its equivalent martingale measure Q_t^T.
  - The arbitrage-free price of the previously defined digital call at time t is
$$DC_t^{D<U}(K,T) = B_t(T)Q_t^T(T_D(K) \le T_U(K) \wedge T).$$
  - In simple models (e.g., constant parameters market model), the previous work computes the price at time 0.
- The contribution of the work: develop replication strategy to hedge the risk of the above digital call.

The Laplace Transform Approach

- Laplace transform (FFT) pricing formula

$$E_{S_0}^{\mathbb{Q}^T}\left[e^{-\lambda T_D(K)} \cdot 1(T_D(K) \leq T_U(K))\right] = \int_{S_0}^{(S_0+K)^-} f(H-K,H,\lambda)dH,$$

$$\text{where } f(L,H,\lambda) = \frac{\partial}{\partial H} E_{S_0}^{\mathbb{Q}^T}\left[e^{-\lambda \tau_L^S} \cdot 1(\tau_L^S \leq \tau_H^S)\right] \text{ for } L < H.$$

- Back to time domain

$$\mathbb{Q}_{S_0}^T\{T_D(K) \leq T_U(K) \wedge T\}$$
$$= \int_{S_0}^{(S_0+K)^-} \frac{\partial}{\partial H} \mathbb{Q}_{S_0}^T\{\tau_{H-K} < \tau_H \wedge T\}dH.$$

- What about the replication at $t > 0$?

Model-free Decomposition

- Let X_t, M_t and m_t be spot price, the historical high and the historical low at time $t \in [0,T]$ of the underlying, respectively.
- On any path in the event $\{T_D(K) \le T_U(K) \wedge T\}$, at $t < T_D(K) \wedge T_U(K)$,
  - If the spot does not reach a new high by $T_D(K)$, $M_{T_D(K)} = M_t$.
  - Otherwise, $M_{T_D(K)} \in (M_t, m_t + K)$.
- Replicate payoff based on the historical high when there is a crash: $M_{T_D(K)}$

$$
\begin{aligned}
&\mathbb{I}_{\{T_D(K) \le T_U(K) \wedge T\}} \\
=&\mathbb{I}_{\{T_D(K) = \tau_{M_{T_D(K)}-K} \le T, M_{T_D(K)} \in [M_t, m_t+K)\}} \\
=&\mathbb{I}_{\{\tau_{M_t-K} \le T, M_{\tau_{M_t-K}} = M_t\}} \\
&+ \int_{M_t^+}^{(m_t+K)^-} \mathbb{I}_{\{\tau_{H-K} \le T\}} \delta(M_{\tau_{H-K}} - H) dH.
\end{aligned}
$$

- Find instruments with desired payoffs.

Hedging instruments

- An one-touch knockout is a double barrier digital option with a (low) in-barrier L and a (high) out-barrier H, the price of this options at time t before its maturity date T is

$$\begin{aligned} OTKO_t(L,H,T) =& B_t(T)Q_t^T(\tau_L \le \tau_H \wedge T) \\ =& B_t(T)Q_t^T(\tau_L \le T, M_{\tau_L} < H). \end{aligned}$$

- The payoff indicator of an one-touch knockout can be modified

$$OTKO_t(L,H^+,T) = B_t(T)Q_t^T(\tau_L \le T, M_{\tau_L} \le H).$$

- A touch-upper-first down-and-in claim is a spread of one-touch knockouts. It has a low barrier L and a high barrier H.

$$\begin{aligned} TUFDI_t(L,H,T) =& \lim_{\varepsilon \to 0^+} \frac{OTKO_t(L,H+\varepsilon,T) - OTKO_t(L,H,T)}{\varepsilon} \\ =& B_t(T)E_t^{Q^T}[\mathbb{1}_{\{\tau_L \le T\}}\delta(M_{\tau_L} - H)], \end{aligned}$$

which pays one dollar at expiry if and only if the spot touches the upper barrier H and then hits L from above before T.

Semi-static replication of one-touch knockouts

- Although the previous replication is fairly robust (no model assumption), the instruments used are rather exotic.
- Under skip-freedom and symmetry assumption, any one-touch knockout can be replicated by single barrier one-touch options: $OT_t(B,T) = B_t(T)Q_t^T(\tau_B \leq T)$.
- We assume that the barriers of an one-touch knockout are skip-free, and when X exit the corridor (L,H), the risk-neutral probability of hitting $X_t - \Delta$ before T is the same as the risk-neutral probability of hitting $X_t + \Delta$ before T, for any $\Delta \geq 0$. This is satisfied by $dX_t = \alpha_t dW_t$ with $d\alpha_t dW_t = 0$.
- Then, ...

Replication of One-touch Knockouts under Arithmetic Symmetry

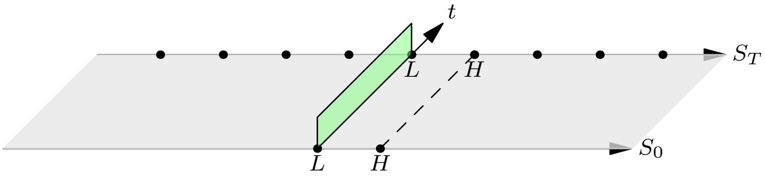
We can show that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L,H,T) = \sum_{n=0}^{\infty} \left\{ OT_t(H-(2n+1)\triangle, T) - OT_t(H+(2n+1)\triangle, T) \right\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits L first,



Replication of One-touch Knockouts under Arithmetic Symmetry

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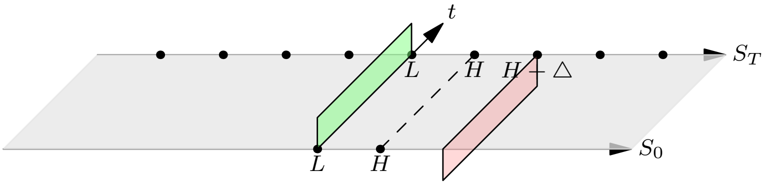
where $\triangle = H - L$.

A sketched proof.

If the spot hits H first,



Replication of One-touch Knockouts under Arithmetic Symmetry

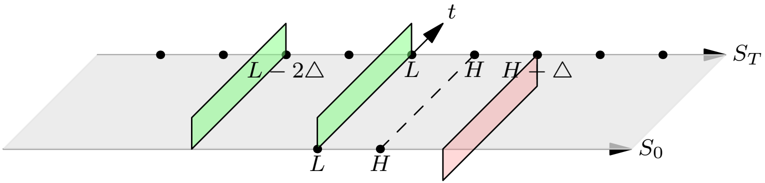
We can show that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L,H,T) = \sum_{n=0}^{\infty} \left\{ OT_t(H-(2n+1)\triangle, T) - OT_t(H+(2n+1)\triangle, T) \right\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits L first,



Replication of One-touch Knockouts under Arithmetic Symmetry

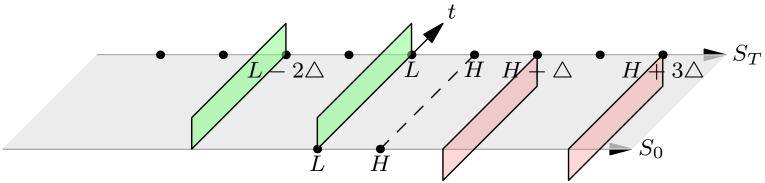
We can show that at $t \in [0, \tau_L \wedge \tau_H \wedge T]$

$$OTKO_t(L,H,T) = \sum_{n=0}^{\infty} \Big\{ OT_t(H-(2n+1)\triangle, T) - OT_t(H+(2n+1)\triangle, T) \Big\},$$

where $\triangle = H - L$.

A sketched proof.

If the spot hits H first,



Semi-Robust Replication of Digital Call on Maximum Drawdown (Carr)

- The maximum drawdown $MD_T = \sup_{s\in[0,T]} DD_s$, is commonly used as a measure of the risk of holding the underlying asset over a period $[0,T]$.
- A risk adverse investor or a portfolio manager can get protection against a loss from the market if he or she holds a claim which pays $1(MD_T \geq K)$, for some strike $K > 0$.
- The maximum drawdown and the maximum drawup over a period $[0,T]$ are related to two stopping times: for $K > 0$

$$T_D(K) = \inf\{t \geq 0, DD_t \geq K\},\ T_U(K) = \inf\{t \geq 0, DU_t \geq K\}.$$

Let us denote by $MU_T = \sup_{s\in[0,T]} DU_s$, then

$$\{MD_T \geq K\} = \{T_D(K) \leq T\},\ \{MU_T \geq K\} = \{T_U(K) \leq T\}.$$

- Introduce another digital call for $K > 0$:
- Digital call on maximum drawdown

$$DC_t^{MD}(K,T) = B_t(T)\mathbb{Q}_t^T\{T_D(K) \leq T\}.$$

Replication of Digital Call on Maximum Drawdown (Carr)

- Under the above **CAHS** assumption, a digital call on maximum drawdown can be replicated with double-one-touches (DOT):

$$DC_t^{MD}(K,T) := B_t(T)1_{\{T_D(K)\le t\}} + 1_{\{T_D(K)>t\}}DOT_t(M_t-K, M_t+K, T).$$

- A double-one-touch is a double barrier digital option with a high barrier H and a low barrier L, the price of this option at time t before its maturity date T is

$$DOT_t(L,H,T) = B_t(T)\mathbb{Q}_t^T\{\tau_L^S \wedge \tau_H^S \le T\}.$$

- In the Bachelier model, using Lévy isomorphism, we have

$$\sup_{s\in[0,t]} W_s - W_t \overset{law}{=} |W_t| \Rightarrow \sup_{t\in[0,T]}\left(\sup_{s\in[0,t]} W_s - W_t\right) \overset{law}{=} \sup_{t\in[0,T]} |W_t|,$$

where W is a standard Brownian motion starting at 0.
- A double-one-touch can be replicated with two one-touch knockouts:

$$DOT_t(L,H,T) = OTKO_t(L,H,T) + OTKO_t(H,L,T).$$

Remark

- The payoff of a digital call on the drawdown of K preceding the drawup of equal size can be semi-statically replicated with one-touches under arithmetic symmetry assumption.
- The replication can also be done with vanilla options (payoff only depends on the value of the stock at maturity). This is the reflection principle: If $H > M_t$

$$OT_t(H,T) = B_t(T)Q_t^T(\tau_H \leq T) = 2B_t(T)Q_t^T(S_T \geq H).$$

- We also developed replicating strategies under geometric symmetry. In particular, under the Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and its independent time-changes S_{β_t} (β_t is a continuous increasing process and $d\beta_t dW_t = 0$), the replicating strategies work well.

Thanks to

- Hongzhong Zhang*
- Peter Carr
- Libor Pospisil
- Jan Vecer

Reference

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  - Zhang, H., Hadjiliadis, O.: Formulas for the Laplace transform of stopping times based on drawdowns and drawups, submitted to *Annals of Applied Probability* on 04-24-09, revised on 02-18-10.
- Maximum drawdown protection:
  - Carr, P., Zhang, H., Hadjiliadis, O.: Insuring against maximum drawdown and drawing down before drawing up, to be submitted to *Finance and Stochastics*.

Summary

- We study drawdown and drawup processes in this work.
- The probability that a drawdown of size a precedes a drawup of size b is fully characterized for biased simple random walk, drifted Brownian motion and more general linear diffusion with continuous generator coefficients.
- Digital insurance can be considered in terms of drawdowns and drawups. Pricing can be done analytically for classical models.
- Robust and semi-robust replicating strategies of the digital insurance are developed. We considered both the arithmetic symmetry and the more involved geometric symmetry in the paper. These strategies are robust to independent continuous time-changes.
- The drawup of log-likelihood ratio process has optimal property when used as a means of detecting abrupt changes.
- We proved the asymptotic optimality of N-CUSUM stopping rule in the multi-source observation setting. In the paper we considered both the Brownian motion system and the discrete-time observation system.